Note: this solution gives the derivation of the correct Hamiltonian intended for question 1 of chapter 2

We want to build the Hamiltonian $H = \mathbf{U}(\mathbf{z})H_0^1\mathbf{U}^\dagger(\mathbf{z})$ from the Hamiltonian

$$H_0^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

From the definition of $\mathbf{U}(\mathbf{z}) = U_1(z_2)U_2(z_2)$ where $U_\alpha(z_\alpha) = \exp(z_\alpha |\alpha\rangle \langle\tilde{2}| - \bar{z_\alpha} |\tilde{2}\rangle \langle\alpha|)$, we can derive an expression for $\mathbf{U}(\mathbf{z})$ (and hence for $\mathbf{U}^\dagger(\mathbf{z})$). Firstly we note that $z_\alpha = \theta_\alpha \exp(i\phi_\alpha)$ and then express $U_\alpha(z_\alpha)$ in terms of sin and cos (which we can do because $U_\alpha(z_\alpha)$ is a complex exponential) for $\alpha = 1, 2$ to give

$$U_1(z_1) = |2\rangle\langle 2| + cos(\theta_1)\big[\,|1\rangle\langle 1| + |\tilde{2}\rangle\langle\tilde{2}|\,\big] + sin(\theta_1)\big[e^{i\phi_1}|1\rangle\langle\tilde{2}| - e^{i\phi_1}|\tilde{2}\rangle\langle 1|\,\big], \tag{2}$$

$$U_2(z_2) = |1\rangle\langle 1| + cos(\theta_2)\big[\,|2\rangle\langle 2| + |\tilde{2}\rangle\langle\tilde{2}|\,\big] + sin(\theta_2)\big[e^{i\phi_2}|2\rangle\langle\tilde{2}| - e^{i\phi_2}|\tilde{2}\rangle\langle 2|\,\big]. \tag{3}$$

Then by working in the basis

$$\big\{\, |1\rangle, |2\rangle, |\tilde{2}\rangle \,\big\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \tag{4}$$

we can express equations 1 and 2 in forms of matrices as all the different ket-bra operators will form 3x3 matrices which can then simply be added together to give

$$U_1(z_1) = \begin{pmatrix} cos(\theta_1) & 0 & e^{i\phi_1}sin(\theta_1) \\ 0 & 1 & 0 \\ -e^{i\phi_1}sin(\theta_1) & 0 & cos(\theta_1) \end{pmatrix}, \tag{5}$$

$$U_2(z_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_2) & e^{i\phi_2}sin(\theta_2) \\ 0 & -e^{i\phi_2}sin(\theta_2) & cos(\theta_2) \end{pmatrix}. \tag{6}$$

We can then calculate

$$\mathbf{U}(\mathbf{z}) = U_1(z_2)U_2(z_2) = \begin{pmatrix} cos(\theta_1) & 0 & e^{i\phi_1}sin(\theta_1) \\ 0 & 1 & 0 \\ -e^{i\phi_1}sin(\theta_1) & 0 & cos(\theta_1) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_2) & e^{i\phi_2}sin(\theta_2) \\ 0 & -e^{i\phi_2}sin(\theta_2) & cos(\theta_2) \end{pmatrix}, \tag{7}$$

$$\therefore \mathbf{U}(\mathbf{z}) = \begin{pmatrix} cos(\theta_1) & -e^{i(\phi_1+\phi_2)}sin(\theta_1)sin(\theta_2) & e^{i\phi_1}sin(\theta_1)cos(\theta_2) \\ 0 & cos(\theta_2) & e^{i\phi_2}sin\theta_2 \\ -e^{i\phi_1}sin(\theta_1) & -e^{i\phi_2}sin(\theta_2)cos(\theta_1) & cos(\theta_1)cos(\theta_2) \end{pmatrix}, \tag{8}$$

which implies that

$$\mathbf{U}^\dagger(\mathbf{z}) = (\bar{\mathbf{U}}(\mathbf{z}))^T = \begin{pmatrix} cos(\theta_1) & 0 & -e^{-i\phi_1}sin(\theta_1) \\ -e^{-i(\phi_1+\phi_2)}sin(\theta_1)sin(\theta_2) & cos(\theta_2) & -e^{-i\phi_2}sin(\theta_2)cos(\theta_1) \\ e^{-i\phi_1}sin(\theta_1)cos(\theta_2) & e^{-i\phi_2}sin\theta_2 & cos(\theta_1)cos(\theta_2) \end{pmatrix}. \tag{9}$$

We can now calculate our transformed Hamiltonian in two steps, the first of which is

$$H_0^1\mathbf{U}^\dagger(\mathbf{z}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} cos(\theta_1) & 0 & -e^{-i\phi_1}sin(\theta_1) \\ -e^{-i(\phi_1+\phi_2)}sin(\theta_1)sin(\theta_2) & cos(\theta_2) & -e^{-i\phi_2}sin(\theta_2)cos(\theta_1) \\ e^{-i\phi_1}sin(\theta_1)cos(\theta_2) & e^{-i\phi_2}sin\theta_2 & cos(\theta_1)cos(\theta_2) \end{pmatrix}. \tag{10}$$

$$\therefore H_0^1\mathbf{U}^\dagger(\mathbf{z}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-i\phi_1}sin(\theta_1)cos(\theta_2) & e^{-i\phi_2}sin\theta_2 & cos(\theta_1)cos(\theta_2) \end{pmatrix}. \tag{11}$$

Our final step is to calculate

$$H = \mathbf{U}(\mathbf{z})H_0^1\mathbf{U}^\dagger(\mathbf{z}) = \mathbf{U}(\mathbf{z})\big(H_0^1\mathbf{U}^\dagger(\mathbf{z})\big), \tag{12}$$

which gives

$$H = \begin{pmatrix} sin^2(\theta_1)cos^2(\theta_2) & e^{i(\phi_1-\phi_2)}sin(\theta_1)sin(\theta_2)cos(\theta_2) & e^{i\phi_1}sin(\theta_1)cos(\theta_1)cos^2(\theta_2) \\ e^{i(\phi_2-\phi_1)}sin(\theta_1)sin(\theta_2)cos(\theta_2) & sin^2(\theta_2) & e^{i\phi_2}sin(\theta_2)cos(\theta_2)cos(\theta_1) \\ e^{-i\phi_1}sin(\theta_1)cos(\theta_1)cos^2(\theta_2) & e^{-i\phi_2}sin(\theta_2)cos(\theta_2)cos(\theta_1) & cos^2(\theta_1)cos^2(\theta_2) \end{pmatrix} \tag{13}$$