1. Firstly, if we have the thermal state

$$\rho = \frac{e^{-H/kT}}{tr(e^{-H/kT})}$$

then we want to expand $e^{-H/kT}$ into something we can work with. By using the hint given in the book we can deduce that when we have $H|\Psi_n\rangle = E_n|\Psi_n\rangle$, the Hamiltonian H can be expanded in terms of its eigenstates to give

$$H = \sum_n E_n |\Psi_n\rangle \langle\Psi_n|$$

This means that $e^{-H/kT}$ becomes

$$e^{-H/kT} = e^{\frac{\sum_n E_n|\Psi_n\rangle\langle\Psi_n|}{kT}} = \sum_n e^{-E_n/kT} |\Psi_n\rangle \langle\Psi_n|$$

This gives

$$\rho = \frac{\sum_n e^{-\beta E_n} |\Psi_n\rangle \langle\Psi_n|}{\sum_n e^{-\beta E_n}}$$

where we have set $1/kT = \beta$. The denominator in the above expression is obtained by noting that $tr(e^{-H/kT})$ is equal to a sum over all of the eigenstate's respective probabilities. We know that $E_{n>0} > E_0$ and we can set $E_0 = 0$. This allows us to rewrite our expression as

$$\rho = \frac{e^{-\beta E_0} |\Psi_0\rangle \langle\Psi_0| + \sum_{n>0} e^{-\beta E_{n>0}} |\Psi_{n>0}\rangle \langle\Psi_{n>0}|}{\sum_n e^{-\beta E_n}}$$

$$\therefore \quad \rho = \frac{|\Psi_0\rangle \langle\Psi_0| + \sum_{n>0} e^{-\beta E_{n>0}} |\Psi_{n>0}\rangle \langle\Psi_{n>0}|}{\sum_n e^{-\beta E_n}}$$

because $e^{-\beta E_0} = e^{-\beta 0} = 1$. Now we want to see what happens to our expression at very low temperatures. To do this we note that

$$\lim_{T\to 0} \rho = \lim_{\beta\to\infty} \rho \quad \because \quad \beta = \frac{1}{kT}.$$

Therefore we can calculate that

$$\lim_{\beta\to\infty} \sum_{n>0} e^{-\beta E_{n>0}} |\Psi_{n>0}\rangle \langle\Psi_{n>0}| = 0 + 0 + 0 + ... = 0$$

and that

$$\lim_{\beta\to\infty} \sum_n e^{-\beta E_n} = 1 + 0 + 0 + ... = 1$$

because $e^{-\beta E_0} = 1$. Finally we can say that

$$\lim_{T\to 0} \rho = \lim_{\beta\to\infty} \rho = \frac{|\Psi_0\rangle \langle\Psi_0| + 0}{1}$$

$$\therefore \quad \lim_{T\to 0} \rho = |\Psi_0\rangle \langle\Psi_0|,$$

which is the ground state as required.

2. The matrix U takes the form

$$\begin{pmatrix} \sigma^x & 0 & \cdots \\ 0 & \sigma^x & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \sigma^x \otimes \mathbb{1}_{2^{n-1}},$$

which allows us to operate on N qubits with a single operation. To simulate this using classical single bit gates we would have to operate on each of our N bits independently.

3. The one-dimensional two qubit cluster state is given by

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}(|{+}0\rangle + |{-}1\rangle).$$

If we measure the second qubit in the $|0\rangle + e^{i\theta}|1\rangle$ basis rotated by $-\pi/2$ (the rotated X-Y plane) then we find that second qubit becomes

$$\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \quad \because \quad e^{-i\pi/2} = -i,$$

as required.

4. We want to transform the state $|\psi\rangle = |00\rangle$ into the maximally entangled state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

when we are in the basis $\{|0\rangle, |1\rangle\}$. Firstly we want to use the Hadamard gate. However, we need a four dimensional Hadamard matrix in order to operate on a two qubit system. We achieve this by taking the tensor product of the two dimensional Hadamard gate with the two dimensional identity matrix

$$\begin{aligned}(H \otimes \mathbb{1}_2)\,|00\rangle &= (H \otimes \mathbb{1}_2)(|0\rangle \otimes |0\rangle)\\ &= H\,|0\rangle + \mathbb{1}_2\,|0\rangle\\ &= (|0\rangle + |1\rangle) \otimes |0\rangle\\ &= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle),\end{aligned}$$

if we assume our state is normalized. We now want to apply a controlled gate that will flip a target qubit iff our control qubit is in the state $|1\rangle$. We want this gate to flip the target qubit so we want to use the controlled-NOT gate which will act as follows

$$\begin{aligned}(CNOT)\big[(H \otimes \mathbb{1}_2)(|0\rangle \otimes |0\rangle)\big] &= (CNOT)(|00\rangle + |10\rangle)\\ &= |00\rangle + |11\rangle\\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\end{aligned}$$

if we assume the normalization condition again. So to transform our state $|00\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, first we apply the Hadamard gates product with the two dimensional identity and then the CNOT gate.