1. (a) $N^c_{1a} = \delta_{ac}$

This property is the trivial statement that particles do not undergo fusion with the vaccum. This is the fundamental property of the vacuum. The δ function states that the only outcome of fusing the particle with the vacuum is to obtain the particle itself, which is the same as having no fusion take place.

(b) $N^1_{ab} = \delta_{b\bar{a}}$

This property states that two particles fuse to the produce vacuum (annihilate), only if they are antiparticles. If particle b annihilates to the vacuum with particle a. Then by definition, this particle is the antiparticle of a, so it can be referred to as b=$\bar{a}$. A particle/antiparticle pair, have the vacuum as their unique fusion channel, thus giving rise to the δ relation.

(c) $N^c_{ab} = N^c_{ba} = N^{\bar{a}}_{b\bar{c}} = N^{\bar{c}}_{\bar{a}\bar{b}}$ b

i. $N^c_{ab} = N^c_{ba}$
The first equality here follows directly from the commutativity property of the fusion rules, specifically that $a \times b = b \times a$. Therefore by changing the order of fusion between two particles, the fusion channel should remain identical.

ii. $N^c_{ab} = N^{\bar{a}}_{b\bar{c}}$

$$\begin{aligned}(a \times b) \times (\bar{c} \times \bar{a}) &= \left(\sum_j N^j_{ab} j\right) \times (\bar{c} \times \bar{a}) = \sum_j N^j_{ab}(j \times \bar{c} \times \bar{a}) \\ &= N^c_{ab}(c \times \bar{c}) \times \bar{a} + \sum_{j \neq c} N^j_{ab}(j \times \bar{c} \times \bar{a}) \\ &= N^c_{ab}\bar{a} + \sum_{j \neq c} N^j_{ab}(j \times \bar{c} \times \bar{a})\end{aligned}$$

$$\begin{aligned}(a \times b) \times (\bar{c} \times \bar{a}) &= (a \times \bar{a}) \times (b \times \bar{c}) \\ &= (1) \times \left(\sum_k N^k_{b\bar{c}} k\right) \\ &= N^{\bar{a}}_{b\bar{c}}\bar{a} + \sum_{k \neq \bar{a}} N^k_{b\bar{c}} k \\ &= N^c_{ab}\bar{a} + \sum_{j \neq c} N^j_{ba}(j \times \bar{c} \times \bar{a})\end{aligned}$$

We would like here to equate the coefficients of the leading $\bar{a}$ terms, however we must first eliminate the possibility of the final fusion term resulting in a $\bar{a}$ anyon. $\sum_{j \neq c} N^j_{ab}(j \times \bar{c} \times \bar{a})$.

By the associativity of the fusion rules, $j \times \bar{c} \times \bar{a} = (j \times \bar{c}) \times \bar{a}$. Due to the fact that anyons do not fuse with the vacuum and change to a different anyon, we have $N^k_{mk} = \delta_{m1}$. Thus for $(j \times c) \times \bar{a}$ to result in a $\bar{a}$ anyon, it must be that $j \times \bar{c} = 1$. As an anyon/anti-anyon pair have a unique fusion channel with the vacuum we have $N^1_m k = \delta_{m\bar{k}}$. Therefore in order for $j \times \bar{c} = 1$, it must be that $j = c$. This is seen to be the original term isolated from the summation, therefore for $j \neq c$, we can not have $j \times \bar{c} \times \bar{a}$ resulting in a $\bar{a}$ anyon.

Finally by equating the coefficients of $\bar{a}$ it is seen that $N^c_{ab} = N^{\bar{a}}_{b\bar{c}}$. As required.

iii. $N^c_{ab} = N^{\bar{a}}_{b\bar{c}}$.
In order to demonstrate this equality we must show that some intermediate results hold. The process will be as follows:

$$N^c_{ab} = N^{\bar{b}}_{a\bar{c}} = N^1_{ab\bar{c}} = N^{\bar{b}c}_a = N^{\bar{a}\bar{b}}_{\bar{c}} = N^{\bar{c}}_{\bar{a}\bar{b}}$$

Firstly we have

$$N_{ab}^{c} = N_{a\bar{c}}^{\bar{b}}$$

from the earlier equality.

The second equality defines what it means to have three anyons undergoing fusion. To show this equality consider:

$$a \times b \times c = \sum_i \sum_j N_{ab}^{i} N_{ic}^{j} j = N_{abc}^{A}$$

Where A is the outcome of fusing all three anyons. In this case we are interested in $A = 1$, the vacuum. We split the above summation to the case $j = 1$ and $j \neq 1$. In the former case, we are forced to take $i = c$, due to the delta relation:

$$N_{ab}^{1} = \delta_{b\bar{a}}$$

wihch dictates that the only way to have the vacuum as outcome from a fusion is to fuse two antiparticles. We then have:

$$N_{ab}^{c} + \sum_{i \neq c} \sum_{j \neq 1} N_{ab}^{i} N_{ic}^{j}$$

The first term here was obtained when we consdiered fusing thee anyons a, b, c to give the vacuum, therefore it can be expressed as $N_{ab\bar{c}}^{1}$, and we have $N_{ab}^{c} = N_{ab\bar{c}}^{1}$.
Next, we consider the composite anyon $b \times \bar{c}$. We are able to manipulate this without evaluating the exact outcome of the fusion. We have that

$$b \times c \times \bar{b} \times \bar{c} = b \times \bar{b} \times c \times \bar{c} = (b \times \bar{b}) \times (c \times \bar{c}) = 1$$

by the commutitivity and associativity properties of fusion. Therefore we know the antiparticle of the $b \times \bar{c}$ anyon is simply $\bar{b} \times c$.
Therefore by the earlier property (ii), we are able to raise the index: $b\bar{c}$, by taking its antiparticle and lowering the 1 anyon (the vacuum) which is its own antiparticle. This gives

$$N_{ab\bar{c}}^{1} = N_{a}^{\bar{b}c}.$$

By applying this property again, to the a, c anyons, we obtain

$$N_{a}^{\bar{b}c} = N_{\bar{c}}^{\bar{b}\bar{a}}$$

The final step is to recall that the fusion process is time invertible as $a \times b = c$ can equivalently be seen as the particle c splitting to give a, b. Therefore we have that:

$$N_{\bar{a}}^{\bar{b}\bar{c}} = N_{\bar{b}\bar{c}}^{\bar{a}}$$

by time inverting the process. And consequently by considering the string of equalities demonstrated:

$$N_{ab}^{c} = N_{a\bar{c}}^{\bar{b}} = N_{ab\bar{c}}^{1} = N_{a}^{\bar{b}c} = N_{\bar{c}}^{\bar{a}\bar{b}} = N_{\bar{a}\bar{b}}^{\bar{c}}$$

we have that:

$$N_{ab}^{c} = N_{\bar{b}\bar{c}}^{\bar{a}}.$$

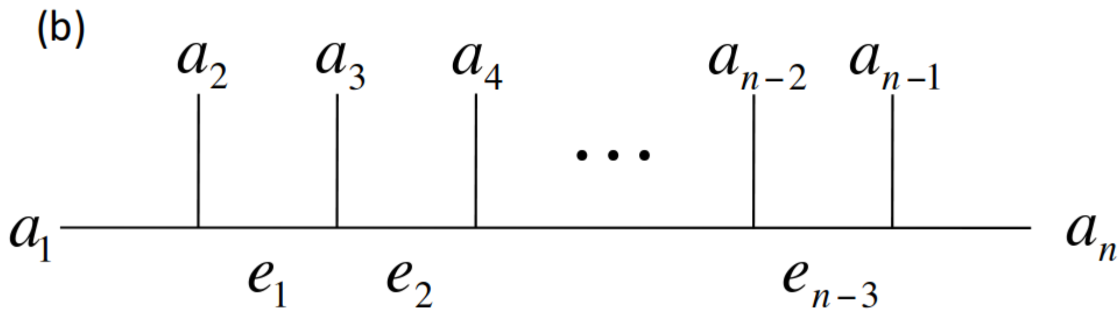
(d)

$$(a \times b) \times c = \left(\sum_d N_{ab}^{e} e \right) \times c = \sum_d \left(\sum_e N_{ab}^{e} N_{ec}^{d} \right) d$$

$$a \times (b \times c) = a \times \sum_f N_{bc}^{f} f = \sum_d N_{bc}^{f} (a \times f)$$
$$= \sum_d \left(\sum_f N_{bc}^{f} N_{af}^{d} \right) d = \sum_d \left(\sum_f N_{af}^{d} N_{bc}^{f} \right) d$$

These fusion rules are associative, meaning $(a \times b) \times c = a \times (b \times c)$, by equating the bracketed coefficients of d in each case it is seen that $\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{bc}^f N_{af}^d$ as required.

2. Consider the fusion of n anyons.



The Hilbert space defining the fusion process will therefore be:

$$\mathcal{H}_{a_1,a_2,\ldots,a_{n-1}}^{e_n} = \bigoplus_{e_1,e_2,\ldots,e_n} \mathcal{H}_{a_1 a_2}^{e_1} \otimes \mathcal{H}_{e_1 a_3}^{e_2} \otimes \mathcal{H}_{e_2 a_4}^{e_3} \otimes \cdots \otimes \mathcal{H}_{a_{n-1} e_{n-3}}^{a_n}$$

This new Hilbert space will have dimension equal to:

$$\dim(\mathcal{M}_n) = \sum_{e_i} N_{a_1 a_2}^{e_1} N_{e_1 a_3}^{e_2} N_{e_2 a_4}^{e_3} \ldots N_{a_{n-1} e_{n-3}}^{a_n}$$

If we consider the matrix N_c with entries $(N_c)_{ab} = N_{ab}^c$, we see that:

$$\dim(\mathcal{M}_n) = \sum_{e_i} N_{a_1 a_2}^{e_1} N_{e_1 a_3}^{e_2} N_{e_2 a_4}^{e_3} \ldots N_{a_{n-1} e_{n-3}}^{a_n} = \left(N_{a_2} N_{a_3} \ldots N_{a_n}\right)_{a_1}^{b_n}$$

We then have

$$\dim(\mathcal{H}_{a_1,a_2,\ldots,a_{n-1}}) = \left(N_{a_2} N_{a_3} \ldots N_{a_n}\right)_{a_1}^{e_n} = \left[N_a^{(n-1)}\right]_a^{e_n} \approx d_a^n$$

3. It is possible to create an entangled state, however it is **not** possible through topologically protected operations alone. More information can be found here:

Brennen, G. K., Iblisdir, S., Pachos, J. K. and Slingerland, J. K. 2009. New J. Phys. 11, 103023

4. Pentagon Equation

F matrix for the Fibonacci model:

$$F_{\tau\tau\tau}^{\tau} = \begin{pmatrix} \frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \\ \frac{1}{\sqrt{\phi}} & -\frac{1}{\phi} \end{pmatrix} \tag{1}$$

Pentagon Equation:

$$(F_{12c}^5)_a^d (F_{a34}^5)_b^c = \sum_e (F_{234}^d)_e^c (F_{1e4}^5)_b^d (F_{123}^b)_a^e \tag{2}$$

The only non-trivial pentagon arises when $1 = 2 = 3 = 4 = 5 = \tau$ [1]. The pentagon equation (2) therefore becomes:

$$(F_{\tau\tau c}^{\tau})_a^d (F_{a\tau\tau}^{\tau})_b^c = \sum_e (F_{\tau\tau\tau}^d)_e^c (F_{\tau e\tau}^{\tau})_b^d (F_{\tau\tau\tau}^b)_a^e$$

Let $a = 1$.

$$(F^{\tau}_{\tau\tau c})^d_1 (F^{\tau}_{1\tau\tau})^c_b = (F^d_{\tau\tau\tau})^c_1 (F^{\tau}_{\tau 1\tau})^d_b (F^b_{\tau\tau\tau})^1_1 + (F^d_{\tau\tau\tau})^c_{\tau} (F^{\tau}_{\tau\tau\tau})^d_b (F^b_{\tau\tau\tau})^{\tau}_1$$

Let $b = 1$.

$$(F^{\tau}_{\tau\tau c})^d_1 (F^{\tau}_{1\tau\tau})^c_1 = (F^d_{\tau\tau\tau})^c_1 (F^{\tau}_{\tau 1\tau})^d_1 (F^1_{\tau\tau\tau})^1_1 + (F^d_{\tau\tau\tau})^c_{\tau} (F^{\tau}_{\tau\tau\tau})^d_1 (F^1_{\tau\tau\tau})^{\tau}_1$$

Here we can identify that $(F^{\tau}_{1\tau\tau})^c_1 = (F^1_{\tau\tau\tau})^{\tau}_1 = (F^{\tau}_{\tau 1\tau})^d_1 = 0$ as they each correspond to an impossible fusion process. The pentagon equation therefore reduces to $0 = 0$ and we move on.

Let $a = 1, b = \tau$.

$$(F^{\tau}_{\tau\tau c})^d_1 (F^{\tau}_{1\tau\tau})^c_{\tau} = (F^d_{\tau\tau\tau})^c_1 (F^{\tau}_{\tau 1\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^1_1 + (F^d_{\tau\tau\tau})^c_{\tau} (F^{\tau}_{\tau\tau\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

Let $a = 1, b = \tau, c = 1$

$$(F^{\tau}_{\tau\tau 1})^d_1 (F^{\tau}_{1\tau\tau})^1_{\tau} = (F^d_{\tau\tau\tau})^1_1 (F^{\tau}_{\tau 1\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^1_1 + (F^d_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

Here $(F^{\tau}_{\tau\tau 1})^d_1 = (F^{\tau}_{1\tau\tau})^1_{\tau} = 0$ as they each corresponds to an impossible fusion process. So we have:

$$0 = (F^d_{\tau\tau\tau})^1_1 (F^{\tau}_{\tau 1\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^1_1 + (F^d_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

Which means:

$$(F^d_{\tau\tau\tau})^1_1 (F^{\tau}_{\tau 1\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^1_1 = -(F^d_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

Let $a = 1, b = \tau, c = 1, d = 1$.

$$(F^1_{\tau\tau\tau})^1_1 (F^{\tau}_{\tau 1\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^1_1 = -(F^1_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

Here we have $(F^{\tau}_{\tau 1\tau})^1_{\tau} = (F^1_{\tau\tau\tau})^1_{\tau} = 0$ as they correspond to impossible fusion processes. The pentagon equation therefore reduces to $0 = 0$ and we move on.

Let $a = 1, b = \tau, c = 1, d = \tau$.

$$(F^{\tau}_{\tau\tau\tau})^1_1 (F^{\tau}_{\tau 1\tau})^{\tau}_{\tau} (F^{\tau}_{\tau\tau\tau})^1_1 = -(F^{\tau}_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

We can see here that $(F^{\tau}_{\tau 1\tau})^{\tau}_{\tau} = 1$ by the fusion diagrams. Upon substitution we arrive at an equation relating elements of the $F^{\tau}_{\tau\tau\tau}$ matrix:

$$(F^{\tau}_{\tau\tau\tau})^1_1 (F^{\tau}_{\tau\tau\tau})^1_1 = -(F^{\tau}_{\tau\tau\tau})^1_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} (F^{\tau}_{\tau\tau\tau})^{\tau}_1 \tag{3}$$

Letting $a = \tau$, the pentagon equation (2) becomes:

$$(F^{\tau}_{\tau\tau c})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^c_b = (F^d_{\tau\tau\tau})^c_1 (F^{\tau}_{\tau 1\tau})^d_b (F^b_{\tau\tau\tau})^1_{\tau} + (F^d_{\tau\tau\tau})^c_{\tau} (F^{\tau}_{\tau\tau\tau})^d_b (F^b_{\tau\tau\tau})^{\tau}_{\tau}$$

Let $a = \tau, b = 1$

$$(F^{\tau}_{\tau\tau c})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^c_1 = (F^d_{\tau\tau\tau})^c_1 (F^{\tau}_{\tau 1\tau})^d_1 (F^1_{\tau\tau\tau})^1_{\tau} + (F^d_{\tau\tau\tau})^c_{\tau} (F^{\tau}_{\tau\tau\tau})^d_1 (F^1_{\tau\tau\tau})^{\tau}_{\tau}$$

We have $(F^{\tau}_{\tau 1\tau})^d_1 = 0$ as it corresponds to an impossible fusion process, also $(F^1_{\tau\tau\tau})^{\tau}_{\tau} = 1$ by the fusion diagram. Therefore:

$$(F^{\tau}_{\tau\tau c})^d_{\tau} (F^{\tau}_{\tau\tau\tau})^c_1 = (F^d_{\tau\tau\tau})^c_{\tau} (F^{\tau}_{\tau\tau\tau})^d_1$$

Let $a = \tau, b = 1, c = 1$

$$(F^{\tau}_{\tau\tau 1})^d_{\tau}(F^{\tau}_{\tau\tau\tau})^1_1 = (F^d_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^d_1$$

Let $a = \tau, b = 1, c = 1, d = 1$

$$(F^{\tau}_{\tau\tau 1})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^1_1 = (F^1_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^1_1$$

Here we see that $(F^{\tau}_{\tau\tau 1})^1_{\tau} = (F^1_{\tau\tau\tau})^1_{\tau} = 0$ as they correspond to impossible fusion processes. Thus the pentagon equation reduces to 0=0 and we move on.

Let $a = \tau, b = 1, c = 1, d = \tau$

$$(F^{\tau}_{\tau\tau 1})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^1_1 = (F^{\tau}_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

We can identify here that $(F^{\tau}_{\tau\tau 1})^{\tau}_{\tau} = 1$ by the fusion diagrams, therefore we obtain another equation relating elements of the $F^{\tau}_{\tau\tau\tau}$ matrix.

$$(F^{\tau}_{\tau\tau\tau})^1_1 = (F^{\tau}_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_1 \tag{4}$$

Consider $a = \tau, b = 1, c = \tau$

$$(F^{\tau}_{\tau\tau\tau})^d_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_1 = (F^d_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^d_1$$

Let $d = 1$

$$(F^{\tau}_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_1 = (F^1_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^1_1$$

Identifying $(F^1_{\tau\tau\tau})^{\tau}_{\tau} = 1$ we see that we have reached the same equation as above (4).

Let $d = \tau$

$$(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_1 = (F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_1$$

This is a trivial equality and of no further use.

Let $a = \tau, b = \tau, c = 1$

$$(F^{\tau}_{\tau\tau 1})^d_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau} = (F^d_{\tau\tau\tau})^1_1(F^{\tau}_{\tau 1\tau})^d_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau} + (F^d_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^d_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}$$

Let $d = 1$

$$(F^{\tau}_{\tau\tau 1})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau} = (F^1_{\tau\tau\tau})^1_1(F^{\tau}_{\tau 1\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau} + (F^1_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}$$

We can see here that $(F^{\tau}_{\tau\tau 1})^1_{\tau} = (F^{\tau}_{\tau 1\tau})^1_{\tau} = (F^1_{\tau\tau\tau})^1_{\tau} = 0$ as they each correspond to an impossible fusion process. The pentagon equation therefore reduces to 0=0 and we move on.

Let $d = \tau$

$$(F^{\tau}_{\tau\tau 1})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau} = (F^{\tau}_{\tau\tau\tau})^1_1(F^{\tau}_{\tau 1\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^1_{\tau} + (F^{\tau}_{\tau\tau\tau})^1_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}$$

Here we have $(F^{\tau}_{\tau\tau 1})^{\tau}_{\tau} = (F^{\tau}_{\tau 1\tau})^{\tau}_{\tau} = 1$ by the fusion diagrams. Upon substitution we arrive at a third equation relating elements of the $F^{\tau}_{\tau\tau\tau}$) matrix:

$$(F^{\tau}_{\tau\tau\tau})^{1}_{\tau} = (F^{\tau}_{\tau\tau\tau})^{1}_{1}(F^{\tau}_{\tau\tau\tau})^{1}_{\tau} + (F^{\tau}_{\tau\tau\tau})^{1}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} \tag{5}$$

Let $a = b = c = \tau, d = 1$

$$(F^{\tau}_{\tau\tau\tau})^{1}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} = (F^{\tau}_{\tau\tau\tau})^{\tau}_{1}(F^{\tau}_{\tau 1\tau})^{1}_{\tau}(F^{\tau}_{\tau\tau\tau})^{1}_{\tau} + (F^{1}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{1}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}$$

Here $(F^{1}_{\tau 1\tau})^{1}_{\tau} = 0$ and $(F^{1}_{\tau\tau\tau})^{\tau}_{\tau} = 1$, giving:

$$(F^{\tau}_{\tau\tau\tau})^{1}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} = (F^{\tau}_{\tau\tau\tau})^{1}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}$$

which is trivial and of no further use.

The final case: $a = b = c = d = \tau$:

$$(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} = (F^{\tau}_{\tau\tau\tau})^{\tau}_{1}(F^{\tau}_{\tau 1\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{1}_{\tau} + (F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}$$

Here $(F^{\tau}_{\tau 1\tau})^{\tau}_{\tau} = 1$ by the fusion rules, giving one final equation relating the elements of the $F^{\tau}_{\tau\tau\tau}$ matrix:

$$(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} = (F^{\tau}_{\tau\tau\tau})^{\tau}_{1}(F^{\tau}_{\tau\tau\tau})^{1}_{\tau} + (F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau} \tag{6}$$

Let us now consider the given $F^{\tau}_{\tau\tau\tau}$ matrix for the Fibonacci model (1).

Subsituting this into the first equation we derived (3), we see that:

$$\frac{1}{\phi} \cdot \frac{1}{\phi} = -\left(\frac{1}{\sqrt{\phi}} \cdot \frac{-1}{\phi} \cdot \frac{1}{\sqrt{\phi}}\right)$$

It is seen that this equation is satisfied for all ϕ.

Substituting (1) into the second equation we derived (4) gives:

$$\frac{1}{\phi} = \frac{1}{\sqrt{\phi}} \cdot \frac{1}{\sqrt{\phi}}$$

This equation is satisfied for all ϕ.

Substituting (1) into the third equation we derived (5) gives:

$$\frac{1}{\sqrt{\phi}} = \left(\frac{1}{\phi} \cdot \frac{1}{\sqrt{\phi}}\right) + \left(\frac{1}{\sqrt{\phi}} \cdot \frac{-1}{\phi} \cdot \frac{-1}{\phi}\right)$$

This is satisfied for $\phi = \frac{1+\sqrt{5}}{2}$: the golden ratio.

Substituting (1) into the final equation we derived (6) gives:

$$\frac{1}{\phi^2} = \frac{1}{\phi} + \left(\frac{-1}{\phi} \cdot \frac{-1}{\phi} \cdot \frac{-1}{\phi}\right)$$

This is satisfied for $\phi = \frac{1+\sqrt{5}}{2}$: the golden ratio.

Thus the given F matrix for the Fibonacci model satisfies the Pentagon equation.

Hexagon Equation

R matrix for the Fibonacci model:

$$R_{\tau\tau} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix} \tag{7}$$

Hexagon Equation:

$$\sum_b (F^4_{231})^c_b R^4_{1b} (F^4_{123})^b_a = R^c_{13} (F^4_{213})^c_1 R^a_{12} \tag{8}$$

As in the pentagon equation, set $1 = 2 = 3 = 4 = 5 = \tau$. The hexagon equation (8) becomes:

$$(F^\tau_{\tau\tau\tau})^c_\tau R^\tau_{\tau\tau} (F^\tau_{\tau\tau\tau})^\tau_a + (F^\tau_{\tau\tau\tau})^c_1 R^\tau_{\tau 1} (F^\tau_{\tau\tau\tau})^1_a = R^c_{\tau\tau} (F^\tau_{\tau\tau\tau})^c_a R^a_{\tau\tau} \tag{9}$$

There are four separate cases for all pairs $(a, c) \in \{1, \tau\}^2$ which must be considered.

(a) $a = c = 1$

$$(F^\tau_{\tau\tau\tau})^1_\tau R^\tau_{\tau\tau} (F^\tau_{\tau\tau\tau})^\tau_1 + (F^\tau_{\tau\tau\tau})^1_1 R^\tau_{\tau 1} (F^\tau_{\tau\tau\tau})^1_1 = R^1_{\tau\tau} (F^\tau_{\tau\tau\tau})^1_1 R^1_{\tau\tau}$$

It is possible to identify that $R^\tau_{\tau 1} = 1$.
Substituting from the $F^\tau_{\tau\tau\tau}$ matrix we obtain an equation relating the components of the $R_{\tau\tau}$ matrix.

$$\frac{1}{\phi^2} + \frac{1}{\phi} R^\tau_{\tau\tau} = \frac{1}{\phi} R^1_{\tau\tau} \cdot R^1_{\tau\tau} \tag{10}$$

(b) $a = \tau, c = 1$

$$(F^\tau_{\tau\tau\tau})^1_\tau R^\tau_{\tau\tau} (F^\tau_{\tau\tau\tau})^\tau_\tau + (F^\tau_{\tau\tau\tau})^1_1 R^\tau_{\tau 1} (F^\tau_{\tau\tau\tau})^1_\tau = R^1_{\tau\tau} (F^\tau_{\tau\tau\tau})^1_\tau R^\tau_{\tau\tau}$$

It is possible to identify that $R^\tau_{\tau 1} = 1$.
Substituting from the $F^\tau_{\tau\tau\tau}$ matrix we obtain a second equation relating the components of the $R_{\tau\tau}$ matrix.

$$\frac{1}{\phi\sqrt{\phi}} - \frac{1}{\phi\sqrt{\phi}} R^\tau_{\tau\tau} = \frac{1}{\sqrt{\phi}} R^1_{\tau\tau} R^\tau_{\tau\tau} \tag{11}$$

(c) $a = 1, c = \tau$

$$(F^\tau_{\tau\tau\tau})^\tau_\tau R^\tau_{\tau\tau} (F^\tau_{\tau\tau\tau})^\tau_1 + (F^\tau_{\tau\tau\tau})^\tau_1 R^\tau_{\tau 1} (F^\tau_{\tau\tau\tau})^1_1 = R^\tau_{\tau\tau} (F^\tau_{\tau\tau\tau})^\tau_1 R^1_{\tau\tau}$$

It is possible to identify that $R^\tau_{\tau 1} = 1$.
Substituting from the $F^\tau_{\tau\tau\tau}$ matrix we obtain a third equation relating the components of the $R_{\tau\tau}$ matrix.

$$\frac{1}{\phi\sqrt{\phi}} - \frac{1}{\phi\sqrt{\phi}} R^\tau_{\tau\tau} = \frac{1}{\sqrt{\phi}} R^1_{\tau\tau} R^\tau_{\tau\tau} \tag{12}$$

This is equal to the second equation obtained (11).

(d) $a=\tau, c=\tau$

$$(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}R^{\tau}_{\tau\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}+(F^{\tau}_{\tau\tau\tau})^{\tau}_{1}R^{\tau}_{\tau 1}(F^{\tau}_{\tau\tau\tau})^{1}_{\tau}=R^{\tau}_{\tau\tau}(F^{\tau}_{\tau\tau\tau})^{\tau}_{\tau}R^{\tau}_{\tau\tau}$$

It is possible to identify that $R^{\tau}_{\tau 1}=1$.
Substituting from the $F^{\tau}_{\tau\tau\tau}$ matrix we obtain a fourth and final equation relating the components of the $R_{\tau\tau}$ matrix.

$$\frac{1}{\phi}+\frac{1}{\phi^2}R^{\tau}_{\tau\tau}=-\frac{1}{\phi}R^{\tau}_{\tau\tau}R^{\tau}_{\tau\tau} \tag{13}$$

We are now in a position to verify that equations (10),(11),(12),(13) hold for the given values of $R^{1}_{\tau\tau}$ and $R^{\tau}_{\tau\tau}$.

Verification

Equation (10)

$$\frac{1}{\phi^2}+\frac{1}{\phi}R^{\tau}_{\tau\tau}=\frac{1}{\phi}R^{1}_{\tau\tau}\cdot R^{1}_{\tau\tau}$$

Substituting from the R matrix (7). We see that:

$$\frac{1}{\phi^2}-\frac{1}{\phi}e^{2\pi i/5}=\frac{1}{\phi}(e^{4\pi i/5})^2$$

$$\implies \frac{1}{\phi}-e^{2\pi i/5}=(e^{4\pi i/5})^2$$
$$\implies \frac{1}{\phi}=e^{2\pi i/5}+e^{8\pi i/5}$$

Convert to polar form and simplify:

$$\begin{aligned} e^{2\pi i/5}+e^{8\pi i/5} &= \cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{8\pi}{5}\right)+i\sin\left(\frac{8\pi}{5}\right)\\ &= \frac{(1-\sqrt{5})}{4}+i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}+\frac{(1-\sqrt{5})}{4}-i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\\ &= \frac{(1-\sqrt{5})}{2}=\frac{2}{1+\sqrt{5}}=\frac{1}{\phi}. \end{aligned}$$

Equation (10) therefore holds for the given R matrix.

Equations (11), (12)

$$\frac{1}{\phi\sqrt{\phi}}-\frac{1}{\phi\sqrt{\phi}}R^{\tau}_{\tau\tau}=\frac{1}{\sqrt{\phi}}R^{1}_{\tau\tau}R^{\tau}_{\tau\tau}$$

$$\begin{aligned} &\frac{1}{\phi}-\frac{-e^{2\pi i/5}}{\phi}=-e^{4\pi i/5}e^{2\pi i/5}=-e^{6\pi i/5}\\ \implies &1-e^{2\pi i/5}=-\phi\cdot e^{6\pi i/5}\\ &\implies \phi=-e^{-6\pi i/5}-e^{-4\pi i/5} \end{aligned}$$

Convert to polar form and simplify:

$$\begin{aligned}\phi &= -e^{-6\pi i/5} - e^{-4\pi i/5} \\ &= -\left(\cos\left(\frac{-6\pi}{5}\right) + i\sin\left(\frac{-6\pi}{5}\right) + \cos\left(\frac{-4\pi}{5}\right) + i\sin\left(\frac{-4\pi}{5}\right)\right) \\ &= -\frac{(-1-\sqrt{5})}{4} - \frac{(-1-\sqrt{5})}{4} + i\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} - i\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} \\ &= -\frac{(-1-\sqrt{5})}{2} = \frac{1+\sqrt{5}}{2}.\end{aligned}$$

Equations (11) and (12) therefore hold for the given R matrix.

Equation (13)

$$\frac{1}{\phi} + \frac{1}{\phi^2}R^{\tau}_{\tau\tau} = -\frac{1}{\phi}R^{\tau}_{\tau\tau}R^{\tau}_{\tau\tau}$$

$$1 - \frac{-e^{2\pi i/5}}{\phi} = -e^{4\pi i/5} \implies \frac{1}{\phi} = \frac{1+e^{4\pi i/5}}{e^{2\pi i/5}} = e^{-2\pi i/5} + e^{2\pi i/5}$$

Convert to polar form and simplify:

$$\begin{aligned}\frac{1}{\phi} &= e^{-2\pi i/5} + e^{2\pi i/5} \\ &= \cos\left(\frac{-2\pi}{5}\right) + i\sin\left(\frac{-2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\ &= \frac{(\sqrt{5}-1)}{4} - i\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} + \frac{(\sqrt{5}-1)}{4} + i\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \\ &= \frac{(\sqrt{5}-1)}{2} = \frac{2}{1+\sqrt{5}} = \frac{1}{\phi}\end{aligned}$$

Equation (13) therefore holds for the given R matrix.

Equations (10) - (13) are each satisfied by the given R matrix for the Fibonacci anyon model. We can therefore conclude that the F and R matrices for the Fibonacci anyon model are consistent with the Pentagon and Hexagon equations.

$$\begin{aligned}\sigma \times \psi &= \sigma \\ \psi \times \psi &= 1 \\ \sigma \times \sigma &= 1 + \psi\end{aligned}$$

References

[1] Trebst et al. (2008). A Short Introduction to Fibonacci Anyon Models. Progress of Theoretical Physics Supplement . 176