5.1.**Question:**Demonstrate explicitly that the $A(\nu)$ and B(p) operators, defined in (5.21) and (5.22) respectively, commute with each other for any v and p.

Answer: We want to demonstrate that $A(\nu)B(p) = B(p)A(\nu)$, where

$$A(\nu) = \frac{1}{|G|}\sum_{g\in G} L^g_{+,1}L^g_{+,2}, L^g_{-,3}L^g_{-,4}, \quad B(p) = \sum_{h_1\ldots h_4=1} T^{h_1}_{-,1}T^{h_2}_{-,2}T^{h_3}_{+,3}T^{h_4}_{+,4}.$$

we want to use the commutation relations (5.20)

$$L^g_+T^h_+ = T^{gh}_+L^g_+, \quad L^g_-T^h_+ = T^{hg^{-1}}_+L^g_-, \quad L^g_+T^h_- = T^{hg^{-1}}_-L^g_+, \quad L^g_-T^h_- = T^{gh}_-L^g_-.$$

The product of $A(\nu)$ and $B(p)$ gives

$$A(\nu)B(p) = \frac{1}{|G|}\sum_{g\in G}\sum_{h_1\ldots h_4=1} L^g_{+,1}L^g_{+,2}L^g_{-,3}L^g_{-,4}T^{h_1}_{-,1}T^{h_2}_{-,2}T^{h_3}_{+,3}T^{h_4}_{+,4}.$$

We want now to see how $L^g_{\pm,i}$ and $T^{h_i}_{\pm,i}$ (i=1,...,4) commute. We have

$$L^g_{-,3}L^g_{-,4}T^{h_1}_{-,1} = L^g_{-,3}T^{gh_1}_{-,1}L^g_{-,4} = T^{g^2h_1}_{-,1}L^g_{-,3}L^g_{-,4}$$

$$L^g_{+,1}L^g_{+,2}T^{g^2h_1}_{-,1} = L^g_{+,1}T^{g^2h_1g^{-1}}_{-,1}L^g_{+,2} = T^{g^2h_1g^{-2}}_{-,1}L^g_{+,1}L^g_{+,2} = T^{h_1}_{-,1}L^g_{+,1}L^g_{+,2}.$$

Similarly we can see how $T^{h_2}_{-,2}$ commutes with $L^g_{\pm,i}$. We have

$$L^g_{-,3}L^g_{-,4}T^{h_3}_{+,3} = L^g_{-,3}T^{h_3g^{-1}}_{+,3}L^g_{-,4} = T^{h_3g^{-2}}_{+,3}L^g_{-,3}L^g_{-,4}$$

$$L^g_{+,1}L^g_{+,2}T^{h_3g^{-2}}_{+,3} = L^g_{+,1}T^{gh_3g^{-2}}_{+,3}L^g_{+,2} = T^{g^2h_3g^{-2}}_{+,3}L^g_{+,1}L^g_{+,2} = T^{h_3}_{+,3}L^g_{+,1}L^g_{+,2}.$$

Similarly we can see how $T^{h_4}_{+,4}$ commutes with $L^g_{\pm,i}$, so that

$$A(\nu)B(p) = \frac{1}{|G|}\sum_{g\in G}\sum_{h_1\ldots h_4=1} T^{h_1}_{-,1}T^{h_2}_{-,2}T^{h_3}_{+,3}T^{h_4}_{+,4}L^g_{+,1}L^g_{+,2}L^g_{-,3}L^g_{-,4} = B(p)A(\nu)$$

5.2.**Question:**Develop explicitly the quantum double theory for the Z_2 group. The D(Z_2) model is the toric code.

Answer: We will follow section 5.3 for an Abelian quantum double model. The elements of the group is

$$\mathbb{Z}_2 = \{0, 1\}$$

The product of the elements of group is

$$0\cdot 0 = 0+0(mod \quad 2) = 0, \quad 1\cdot 1 = 1+1(mod \quad 2) = 1, \quad 1\cdot 0 = 0+1(mod \quad 2) = 1.$$

The lattice has 2-level spins on every edge. Rotations are in term of the Pauli operators

$$X = |0+1(mod \quad 2)\rangle\langle 0| + |1+1(mod \quad 2)\rangle\langle 1| = |1\rangle\langle 0| + |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Z = (e^{i\pi})^0(|0\rangle\langle 0|) + (e^{i\pi})^1(|1\rangle\langle 1|) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The commutation relation is satisfied

$$ZX = -XZ$$

The eigenstates are

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}(\omega^{0\cdot 0}|0\rangle + \omega^{0\cdot 1}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\tilde{1}\rangle = \frac{1}{\sqrt{2}}(\omega^{1\cdot 0}|0\rangle + \omega^{1\cdot 1}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

with the corresponding eigenvalues being 0 and 1.

The Hamiltonian is

$$H = -\sum_{\nu}(I + A(\nu)) - \sum_{p}(I + B(p))$$

There are 4 particle species $1, e, m, \epsilon$ with fusion rules

$$e \times e = e, \quad e \times 1 = e, \quad m \times m = m, \quad m \times 1 = m, \quad e \times m = \epsilon$$

Finally, the R matrix of the braiding operation of e around an m is

$$(R^{\epsilon}_{em})^2 = \omega^{1\cdot 1} = -1 \to R^{\epsilon}_{em} = \pm i$$

5.3.**Question:**Establish the fourfold degeneracy of the toric code by demonstrating that non-trivial loops with $C^1_{\sigma^z}$ and $C^2_{\sigma^z}$ operations or combinations of them with $C^1_{\sigma^x}$ and $C^2_{\sigma^x}$ gives states that are linearly dependent to (5.16).

Answer: We want to demonstrate that $\{|0\rangle, C^1_{\sigma^x}|0\rangle, C^2_{\sigma^x}|0\rangle, C^1_{\sigma^x}C^2_{\sigma^x}|0\rangle\}$ is related to $\{|0\rangle, C^1_{\sigma^z}|0\rangle, C^2_{\sigma^z}|0\rangle, C^1_{\sigma^z}C^2_{\sigma^z}|0\rangle\}$, we will denote the states of the first set as

$$ket00_x = |0\rangle, \quad |10\rangle_x = C^1_{\sigma^x}|0\rangle, \quad |01\rangle_x = C^2_{\sigma^x}|0\rangle, \quad |11\rangle_x = C^1_{\sigma^x}C^2_{\sigma^x}|0\rangle \tag{1}$$

and the states of the second set as

$$ket00_z = |0\rangle, \quad |10\rangle_z = C^1_{\sigma^z}|0\rangle, \quad |01\rangle_z = C^2_{\sigma^z}|0\rangle, \quad |11\rangle_z = C^1_{\sigma^z}C^2_{\sigma^z}|0\rangle. \tag{2}$$

As C_{σ^x} loops are defined on the dual lattice of the lattice on which C_{σ^z} are defined , we can think the C operators acting the same way as the Pauli operators do on two qubits and such

$$ket00_z = \frac{1}{2}(|00\rangle_x + |10\rangle_x + |01\rangle_x + |11\rangle_x) = \frac{1}{2}(|00\rangle_x + C^1_{\sigma^x}|00\rangle_x + C^2_{\sigma^x}|00\rangle_x + C^1_{\sigma^x}C^2_{\sigma^x}|00\rangle_x). \tag{3}$$

Similarly, we have

$$|10\rangle_z = \frac{1}{2}(|00\rangle_x - C^1_{\sigma^x}|00\rangle_x + C^2_{\sigma^x}|00\rangle_x - C^1_{\sigma^x}C^2_{\sigma^x}|00\rangle_x)$$

$$|01\rangle_z = \frac{1}{2}(|00\rangle_x + C^1_{\sigma^x}\,|00\rangle_x - C^2_{\sigma^x}\,|00\rangle_x - C^1_{\sigma^x}C^2_{\sigma^x}\,|00\rangle_x)$$

$$|11\rangle_z = \frac{1}{2}(|00\rangle_x - C^1_{\sigma^x}\,|00\rangle_x - C^2_{\sigma^x}\,|00\rangle_x + C^1_{\sigma^x}C^2_{\sigma^x}\,|00\rangle_x).$$

Inverting these relations, we have

$$ket00_x = \frac{1}{2}(|00\rangle_z + |10\rangle_z + |01\rangle_z + |11\rangle_z) = \frac{1}{2}(|00\rangle_z + C^1_{\sigma^z}\,|00\rangle_z + C^2_{\sigma^z}\,|00\rangle_z + C^1_{\sigma^z}C^2_{\sigma^z}\,|00\rangle_x). \quad (4)$$

$$|10\rangle_x = \frac{1}{2}(|00\rangle_z - C^1_{\sigma^z}\,|00\rangle_z + C^2_{\sigma^z}\,|00\rangle_z - C^1_{\sigma^z}C^2_{\sigma^z}\,|00\rangle_z)$$

$$|01\rangle_z = \frac{1}{2}(|00\rangle_z + C^1_{\sigma^z}\,|00\rangle_z - C^2_{\sigma^z}\,|00\rangle_z - C^1_{\sigma^z}C^2_{\sigma^z}\,|00\rangle_z)$$

$$|11\rangle_x = \frac{1}{2}(|00\rangle_z - C^1_{\sigma^z}\,|00\rangle_z - C^2_{\sigma^z}\,|00\rangle_z + C^1_{\sigma^z}C^2_{\sigma^z}\,|00\rangle_z).$$

5.4.**Question:**What is the ground state degeneracy of the toric code model, defined on an infinite plane with punctures in the form of absent vertex or plaquette interaction terms in the Hamiltonian.

Answer: An infinite lattice has open boundaries, we can introduce punctures on its surface to encode information. The punctures can be introduced by measuring the stabilizer, thus disentangling the measured spins from the code.

There are two type of punctures, if a plaquette operator is measrues we create a rough boundary (Figure 1(a)). If a vertex operator is measures we create a smooth boundary (Figure 1(c)).

A logical qubit is encoded by the string of Pauli operators (σ^x or σ^z) which define a logical operator ($\overline{X}$ or $\overline{Z}$), see Figure 1(b) and 1(d).

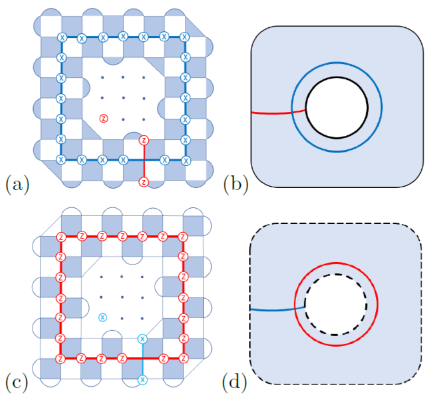

Figure 1: Different type of puncture defects on the toric code. The puncture and the code boundary are (a)rough and (c)smooth. Measured stabilizers (punctures) and non-contractible loops stabilizing them are also shown in (a) and (c). Diagrammatic representation of punctures in (b) and (d).

For more information: https://arxiv.org/abs/2103.08381