7.1

Problem

Demonstrate that the Abelian Chern-Simons action (7.7) is invariant under the gauge transformation

$$A_\mu(x) \longrightarrow A_\mu^\omega(x) = A_\mu(x) + \partial\omega(x) \tag{1}$$

For the derivation assume that A_μ goes to zero at the boundaries of the system.

Solution

The gauge we are applying is

$$A_\mu(x) \longrightarrow A_\mu^\omega(x) = A_\mu(x) + \partial\omega \tag{2}$$

with ω being a scalar field. Applying this gauge to the Chern-Simons action we get

$$I_{CS}^\omega = \frac{m}{2}\int_M d^3x\varepsilon^{\mu\nu\rho}(A_\mu + \partial_\mu\omega)\partial_\nu(A_\rho + \partial_\rho\omega) \tag{3}$$

Implying:

$$I_{CS}^\omega = I_{CS} + \int_M d^3x\varepsilon^{\mu\nu\rho}(\partial_\mu\omega\partial_\nu A_\rho + \partial_\mu\omega\partial_\nu\partial_\rho\omega + A_\mu\partial_\nu\partial_\rho\omega) \tag{4}$$

If we consider the second and fourth term on the RHS we can represent them as a surface integral of a total derivative

$$\partial_\mu\omega\partial_\nu A_\rho + A_\mu\partial_\nu\partial_\rho\omega = \partial_\nu\omega\partial_\rho A_\mu + A_\mu\partial_\nu\partial_\rho\omega = \partial_\rho(A_\mu\partial_\nu\omega) \tag{5}$$

And so this term can be ignored, the third term can be represented as

$$\begin{aligned}\varepsilon^{\mu\nu\rho}\partial_\mu\omega\partial_\nu\partial_\rho\omega = \partial_0\omega(\partial_1\partial_2\omega - \partial_2\partial_1\omega)\\ +\partial_1\omega(\partial_2\partial_0\omega - \partial_0\partial_2\omega) + \partial_2\omega(\partial_0\partial_1\omega - \partial_1\partial_0\omega)\end{aligned} \tag{6}$$

which vanishes as $[\partial_i, \partial_j] = 0$ for all i and j, so finally we get

$$I_{CS}^\omega = I_{CS} \tag{7}$$

7.2

Problem

Consider the Abelian Chern-Simons theory for a given m. Take a source to be a disk of radius R with homogeneous distribution of charge Q on it. Derive the spin (7.14) of the source by calculating the total phase accumulated from the charge when we rotate the disk by 2n. [Hint: Evaluate the phase of a small ring of charge of radius $0 \leq r \leq R$ due to the flux enclosed by the ring].

Solution

It is known that the phase of a ring rotating around an enclosed flux is given by $\phi = q\Phi$, ring charge q, but when considering the effects on a disk, the variation in flux and charge density with respect to the radius of the disk must be taken into consideration, and so

$$2\phi(r) = \rho A(r)\Phi(r), \tag{8}$$

where ρ is the charge density of the disk, Φ how the enclosed flux varies with radius r and A the area of the disk $A(r) = 2\pi r dr$, $2\pi r$ the circumference of the ring and the dr element an extension of the ring into a disk. As the charge is homogeneously distributed, it is just an integration of the charge density over the area the charge occupies. Integrating this ring into a disk:

$$A = \int_r^R A(r) = \int_r^R 2\pi r dr = \pi R^2 - \pi r^2. \tag{9}$$

The flux as found in (7.12) is

$$\Phi = \frac{Q(r)}{m} = \frac{\rho A_T(r)}{m}, \tag{10}$$

where A_T is the area enclosed by the outer surface of the disk (πr^2), as this is the total contained flux of the system, therefore does not depend on the inner geometry of the disk. This is unlike the area A which was the area in which the charge q of the ring was 'stretched' to form the disk of charge, as the area A_T is the area encompassed by the total charge of the disk Q, i.e. the total area contained within the outer radius of the disk. $\Phi(r)$ is the effect of the enclosed flux together with the area it is encompassing. Applying these ideas into the phase we derive

$$2\phi(r) = \rho A\Phi(r) = \rho(\pi R^2 - \pi r^2)\frac{\rho A_T(r)}{m}. \tag{11}$$

For the total phase we integrate over this ring:

$$2\phi = \int_0^R \phi(r) = \int_0^R \rho(\pi R^2 - \pi r^2)\frac{\rho A_T(r)}{m} = \frac{2\pi^2\rho^2}{m}\left[R^2 r^2 - \frac{r^4}{2}\right]_0^R, \tag{12}$$

giving a final phase independent of r:

$$\phi = \frac{\rho^2}{2m}\pi^2 R^4. \tag{13}$$

The area enclosed by the disk is given by radius R, and so $\rho\pi R^2 = Q$, which can be taken twice resulting in

$$\phi = \frac{Q^2}{2m}. \tag{14}$$

It is known a circulation of an anyon around itself is given by a spin phase $s = \frac{\phi}{2\pi}$, and finally we arrive at:

$$s = \frac{Q^2}{4\pi m.} \tag{15}$$

7.3

Problem

Prove that

$$e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}} = \mathcal{A} + \lambda[\mathcal{A},\mathcal{B}], \tag{16}$$

when $[\mathcal{A},[\mathcal{A},\mathcal{B}]] = [\mathcal{B},[\mathcal{B},\mathcal{A}]] = 0$. Then demonstrate that relation (7.71) holds. [Hint: Consider the function $f(x) = e^{x\mathcal{A}}e^{x\mathcal{B}}$ and its differentiation with respect to x].

Solution

Taking the derivative of the function in the hint acting on a lie algebra element we get

$$\frac{d}{d\lambda}(e^{\lambda\mathcal{A}}e^{\lambda\mathcal{B}}) = \mathcal{A}e^{\lambda\mathcal{A}}e^{\lambda\mathcal{B}} + e^{\lambda\mathcal{A}}\mathcal{B}e^{\lambda\mathcal{B}} \tag{17}$$

As $[\mathcal{B}, e^{\lambda\mathcal{B}}] = 0$ these elements can be swapped, applying this derivative to the equation we wish to solve

$$\begin{aligned}\frac{d}{d\lambda}(e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}}) &= -\mathcal{B}e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}} + e^{-\lambda\mathcal{B}}\mathcal{A}\mathcal{B}e^{\lambda\mathcal{B}} \\ &= -\mathcal{B}e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}} + e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}}\mathcal{B}.\end{aligned} \tag{18}$$

If we define an operator $ad_X Y = [X, Y] = XY - YX$, the last equation can be written as $ad_{-\mathcal{B}}(e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}})$, so we find that

$$f'(\lambda) = ad_{-\mathcal{B}}f(\lambda) \tag{19}$$

and as $f(0) = 1$ it is clear that $f(\lambda) = e^{\lambda ad_{-\mathcal{B}}}$. This gives us an alternative form of our original equation

$$e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}} = f(\lambda)\mathcal{A} = e^{\lambda ad_{-\mathcal{B}}}\mathcal{A} \tag{20}$$

Using the expansion of the exponential

$$e^x = 1 + x + \frac{x^2}{2} + ... \tag{21}$$

And finally, we can derive the expression

$$\begin{aligned}e^{-\lambda\mathcal{B}}\mathcal{A}e^{\lambda\mathcal{B}} = e^{\lambda ad_{-\mathcal{B}}}\mathcal{A} &= (1 + \lambda ad_{-\mathcal{B}} + (\lambda ad_{-\mathcal{B}})^2 + ...)\mathcal{A} \\ &= \mathcal{A} + [-\lambda\mathcal{B}, \mathcal{A}] + \frac{1}{2}[-\lambda\mathcal{B}, [-\lambda\mathcal{B}, \mathcal{A}]] + ... \\ &= \mathcal{A} + \lambda[\mathcal{A}, \mathcal{B}]\end{aligned} \tag{22}$$